Rational functions

Let $\emptyset \neq V \subseteq \mathbb{A}^n$ be a variety (i.e. irreducible)

$\Gamma(V)$ is an integral domain $\Rightarrow$ we have the following def:

Def: The field of rational functions on V, denoted $k(V)$, is the field of fractions of $\Gamma(V)$. $f \in k(V)$ is a rational function on V.

Ex: In $V(xy - z^2) \subseteq \mathbb{A}^3$, $\frac{x}{z}$ is the same rational function as $\frac{z}{y}$.

Def: A rational function $f \in k(V)$ is defined or regular at $P \in V$ if $\exists\ g, h \in \Gamma(V)$ s.t. $f = \frac{g}{h}$ and $h(P) \neq 0$.

In the previous example, $f = \frac{x}{z} = \frac{z}{y}$ is defined at (x, y, z) if $z \neq 0$ or $y \neq 0$.

Poles of rational functions

Def: Let $f \in k(V)$, $P \in V$. P is a pole of f if f is not defined at f. (i.e. every possible denominator vanishes at P).

Ex: 1.) If $\Gamma(V)$ is a UFD, then up to multiplication by units, $f \in k(V)$ can be written uniquely as $f = \frac{a}{b}$, where a, b rel. prime, so the pole set of f is $V(b)$.

2.) In the $V(xy-z^2)$ example, the pole set of $\frac{x}{z}=\frac{z}{y}$ is $V(\overline{z},\overline{y}) \subseteq V(xy-z^2)$.

Prop: The set of poles of a rational function is an algebraic subset of V.

Pf: Suppose $V \subseteq \mathbb{A}^n$. Let $f \in k(V)$. Let $J_f = \{ g \in \Gamma(V) \mid gf \in \Gamma(V) \}$

Easy to check: J_f is an ideal.

WTS: $V(J_f) =$ pole set of f.

P is <u>not</u> a pole of $f \iff \exists\ a, b \in \Gamma(V)$ s.t. $\frac{a}{b} = f$, $b(P) \neq 0$
$\iff \exists\ b \in J_f$ s.t. $b(P) \neq 0$.
$\iff P \notin V(J_f)$. $\square$

Local rings at points

Def: Let $P \in V$. $\mathcal{O}_P(V) \subseteq k(V)$ is the set of rational functions on V that are defined at P, called the <u>local ring</u> of V at P.

Caution: $\mathcal{O}_P(V) \not\cong k(P)$.

Ex: $P = V(x) \subseteq \mathbb{A}^1$

$\Gamma(P) = {}^{k[x]}/_{(x)} \cong k$ so $k(P) \cong k$.

However, $\frac{x}{1} \in \mathcal{O}_P(\mathbb{A}^1)$, but $\frac{1}{x} \notin \mathcal{O}_P(\mathbb{A}^2)$, so $\mathcal{O}_P(\mathbb{A}^1)$ is not a field.

Although x evaluated at P is 0, $x \neq 0$ in $\mathcal{O}_P(\mathbb{A}^1)$.

More generally, $\mathcal{O}_P$ depends on V, whereas $k(P) = \Gamma(P) \cong k$, always.

Claim: $\mathcal{O}_P(V)$ is a subring of $k(V)$.

Pf: $\frac{a}{b}, \frac{c}{d} \in \mathcal{O}_P(V)$ s.t. $b(P), d(P) \neq 0$. Then $b(P)d(P) \neq 0$,

so products and differences are in $\mathcal{O}_P(V)$.

So
$$k \subseteq \Gamma(V) \subseteq \mathcal{O}_P(V) \subseteq k(V) \qquad \square$$
$$f \longmapsto f/1$$

Prop: $\Gamma(V) = \bigcap_{P \in V} \mathcal{O}_P(V)$, for V a variety. (i.e. $\Gamma(V)$ is exactly the rat'l functions defined at every point of V)

Pf: We know $\subseteq$. If $f \in \bigcap_{P \in V} \mathcal{O}_P(V)$, then f has no poles,

so if $J_f = \{ g \in \Gamma(V) \mid gf \in \Gamma(V) \}$, $V(J_f) = \emptyset$.

$\Rightarrow 1 \in J_f$ (Weak Nullstellensatz) $\Rightarrow f \in \Gamma(V)$. $\square$

If $f \in \mathcal{O}_P(V)$, we can evaluate at P:

If $f = \frac{a}{b} = \frac{a'}{b'}$ $\Rightarrow a(P)b'(P) = a'(P)b(P) \Rightarrow \frac{a(P)}{b(P)} = \frac{a'(P)}{b'(P)}$.

nonzero on P

i.e. evaluation of f is well-defined.

Evaluation gives us a homomorphism:

$$\mathcal{O}_P(V) \longrightarrow k$$
$$f \longrightarrow f(P)$$

Since $k \subseteq \mathcal{O}_P(V)$ maps to itself, this map is surjective.

The kernel is thus max'l, called the <u>maximal ideal of V at P</u>, defined $m_P(V) = \{\text{non-units of } \mathcal{O}_P(V)\} = \left(\frac{g}{1} \,\middle|\, g \in I_V(P)\right)$

Def/Lemma: A ring R is a <u>local ring</u> if it satisfies the following equivalent conditions:

1.) The set of non-units in R is an ideal.

2.) R has a unique maximal ideal m.

(i.e. $\mathcal{O}_P(V)$ is, in fact, a local ring)

Pf: 1.) $\Rightarrow$ 2.): Let m be the ideal consisting of non-units.
If $I \subsetneq R$ is an ideal, it contains no units, so $I \subseteq m$.

2.) $\Rightarrow$ 1.): Let m be the unique maximal ideal. Then if

$a \in R$ is not a unit, $(a) \subsetneq R$, so $(a) \subseteq m$. Thus m contains all non-units, so it is exactly the set of non-units. $\square$

Ex: 1.) Let $R = \{\frac{a}{b} \mid a, b \in \mathbb{Z},\ b \text{ odd}\}$.

R is a ring (check)

$c \in R$ is a non-unit $\iff$ $c = \frac{2a}{b}$ where b is odd

$\iff c \in (2)$

Thus, R is a local ring.

2.) $\mathbb{C}[x]$ is not a local ring: $x+1$ and x are non-units, but $(x+1) - x = 1$, so the non-units don't form an ideal.

3.) Let $R = \{\frac{a}{b} \in k(x) \mid a, b \in k[x] \text{ and } b \text{ has a nonzero constant term}\}$

Exer: R is a local ring w/ max'l ideal $(\frac{x}{1})$.

In fact: $R = \mathcal{O}_0(\mathbb{A}^1)$.

Prop: $\mathcal{O}_p(V)$ is Noetherian.

Pf: Let $I \subseteq \mathcal{O}_p(V)$. WTS I is finitely generated.

Consider $J = I \cap \Gamma(V)$. J is an ideal of $\Gamma(V)$.

$\Gamma(V)$ is Noetherian, so $J = (f_1, \dots, f_r) \subseteq \Gamma(V)$.

Let $f \in I \subseteq \mathcal{O}_P(V)$. Then $f \in \frac{a}{b}$, $a, b \in \Gamma(V)$, $b(P) \neq 0$.

Thus, $bf = a \in I \cap \Gamma(V) = J$.

$\Rightarrow$ $bf = a_1 f_1 + \dots + a_r f_r$, $a_i \in \Gamma(V)$.

$\Rightarrow$ $f = \left(\frac{a_1}{b}\right) f_1 + \dots + \left(\frac{a_r}{b}\right) f_r \Rightarrow I = (f_1, \dots, f_r)$. $\square$

Let $\varphi: V \longrightarrow W$ be a regular map of affine varieties.

Consider

$$\begin{array}{ccc} \varphi^* : \Gamma(W) & \longrightarrow & \Gamma(V) \\ \downarrow & & \downarrow \\ k(W) & & k(V) \end{array}$$

Q: Can we extend φ^* to $k(W)$?

If so, there's only one possible map:

$$\frac{g}{h} \longmapsto \frac{\varphi^*(g)}{\varphi^*(h)}$$

But if $h \in \ker \varphi^*$, this doesn't work!

(Note: it does work if φ is dominant! Do you see why this is true geometrically?)

Instead, let $P \in V$. Set $Q = \varphi(P)$.

Let $h \in \Gamma(W)$ s.t. $h(Q) \neq 0$.

Then $V \longrightarrow W \xrightarrow{h} k$

$$\begin{tikzcd} P \arrow[r, mapsto] \arrow[rr, bend right, "\varphi^*(h)"'] & Q \arrow[r, mapsto] & h(Q) \neq 0 \end{tikzcd}$$

Thus, if $\frac{g}{h}$ is defined at Q and $h(Q) \neq 0$,

then $\dfrac{\varphi^*(g)}{\varphi^*(h)}$ is defined at P.

In fact, this gives us a well-defined (check!) map, so φ^* induces a morphism

$$\mathcal{O}_Q(W) \longrightarrow \mathcal{O}_P(V).$$

Furthermore, if $\frac{g}{h} \in m_Q(W)$, then $g(Q) = 0$,

so $\varphi^*(g)(P) = g(\varphi(P)) = g(Q) = 0$

so $\varphi^*(\frac{g}{h}) \in m_P(V)$. That is, $m_Q(W)$ gets mapped into $m_P(V)$.